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Effective Fractal Dimension at 2d-3d Crossover

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Abstract: This article is aimed at reviewing and studying the effects of the 2d-3d crossover on the effective fractal and spatial dimensions, as well as on the critical exponents of the physical properties of bulk and bounded systems at criticality. Here we consider the following problems: (1) the two types of dimensional crossovers and the concept of the universality classes; (2) a smooth 2d-3d crossover and the calculation of the effective fractal and spatial dimensions, as well as the effective critical indices; (3) the fractal dimension, its connection with the random mean square order-parameter fluctuations and a new phase formation; (4) the fractal nuclei of a new phase and the medical consequences of carcinogenesis and nucleation isomorphism.

Keywords: fractals; effective fractal dimension; dimensional crossover; critical indices (exponents); bounded systems; nucleation; carcinogenesis

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1. Introduction

The main idea of this article is to study the fractal nature and fractal dimensions of the physical properties of bulk and bounded condensed systems [1–12] from the point of view of the theory of the phase transitions and critical phenomena [13–24]. The influence of confinement on various phase transitions and critical phenomena were actively studied in many systems of experimental, theoretical and practical interest as follows: bounded fluids and liquid crystals, magnetic systems, porous media, vesicles, biomembranes, synaptic clefts, etc. [25–39]. In particular, the consequences of the influence of the 2d-3d crossover (DC) on the change in the fractal dimension will be considered. It should be emphasized that the direct and reverse transition between three-dimensional and two-dimensional systems cannot occur abruptly, but must be associated with a fairly smooth and continuous change in physical properties (for example, such critical parameters as the critical indices of scaling laws or the critical values of temperature, density, pressure, etc.). The basis for this statement is the results of the theoretical studies of the layer-by-layer ordering and the DC in [40–43], as well as the results of computer-simulation studies [28,29,44].

In [41–43], we studied the physical properties of the bounded systems as a result of a DC of the following two types: *the 1st type*–the transition from 3d bulk systems to 3d bounded systems when the linear size L, in the direction of the spatial confinement approaches the correlation radius ξ of the fluctuations of the order parameter (for single-component fluids–density fluctuations); *the 2nd type*–a smooth change in the linear dimensions, which causes a transition between 3d and 2d systems, taking into account the corresponding changes in the fractal and spatial dimensions, as well as the critical indices of the physical properties of the bulk and bounded systems at criticality. In connection with the 2d-3d C, I would like to draw attention to an example that is associated with the definition of such a physical quantity as the surface tension: *the surface tension determines the average energy of the transfer of molecules from the "bulk" state to the "surface" (two-dimensional) state* [45]. In other words, the surface tension coefficient is directly related to the DC of the 2nd type.

Here in this paper, the following findings will be considered in detail: (a) the influence a DC of the 2nd type on the fractal dimension [41], as well as (b) the relationship between

the fractal dimension and the random mean-square (RMS) fluctuations of the number of particles (in a more general case, the fluctuation of the corresponding order parameters in the close vicinity of the critical (bifurcation) points and the phase transition points [13–24]). For this purpose, a synergetic analogy between the physical processes in the metastable state [46–53] (namely, the nucleation of fractal nuclei of a new phase [48]) and the similar process of carcinogenesis (the nucleation of malignant neoplasms) in oncology [54,55]. Obviously, synergistically similar processes of phase transitions will require a more detailed consideration of the concept of "the universality class", which includes systems of a different nature with the same critical (bifurcation) behavior [17–22,41].

The foregoing defines the following structure of this article, being both a review and a publication containing original results. The theoretical background of studying the bounded systems and the classes of universality will be studied and discussed in the Section 2. Next, the Section 3 will be devoted to studying a smooth layer ordering at the 2d-3d crossover and deriving the formula for effective critical indices at the second-order phase transitions and critical phenomena. Here in this Section 3, the results of calculating the effective fractal and spatial dimensions will also be investigated. The main goal of the Section 4 is to find a connection between the fractal dimension and the RMS order-parameter fluctuation, as well as to study the problem of a new fractal phase formation. In the Sections 5 and 6, the fractal nuclei of a new phase and the medical consequences of carcinogenesis and nucleation isomorphism will be discussed.

2. Classes of Universality and the Theoretical Background of Studying the Bounded Systems

The unification of two outstanding achievements in physics of the XXth century, namely: (1) the revolutionary discoveries and the achievements in nanosciences (including nanomedicine), which will determine the future face of the XXIst century, and (2) the solution of problems of the 2nd order phase transitions, that continues to incite the increasing interest of researchers to the study of phase transitions in the mesoscale (nano- and micro-) systems. The ideas of universality and isomorphism of the phase transitions and critical phenomena allow, in particular, to transfer the properties of the confined physical systems to much more complex systems of a medical nature [17–22,24,31–35].

The concept of *universality classes* becomes very important for this purpose. The basic conditions of the universal behaviour for infinite-size (bulk) systems of a different nature are well-known [13,17–22]: (1) the same space dimensionality; (2) the same dimensionality (number of components) of the order parameters; (3) the same type of intermolecular interaction (short- or long-range); (4) the same symmetry of a Hamiltonian (fluctuation part of the thermodynamic potential). The universal behaviour for the confined (bounded) systems needs the following conditions, in addition to four previous conditions: (5) the same geometric factors (number of monolayers) for the system's confinement; (6) the same low crossover dimensionality (geometric form of the restricted volume); (7) the same type of boundary conditions; (8) the same physical properties under consideration [25,27,41,42].

The essential progress in physics of the phase transitions and the critical phenomena of various nature are achieved because of a deeper understanding of the role of the correlation effects of the order-parameter fluctuations at large distances and times. The creation of the theory of superconductivity, by Vitaliy Ginzburg and Lev Landau [14], the development of fundamental ideas of the fluctuation theory of scaling invariance [17–20] and the renormalization group approach by Kenneth Wilson and Michel Fisher [21,22], understanding the Levanyuk–Ginsburg conditions under which the Landau mean-field theory of phase transitions is valid [15,16], and a number of precise experiments (see, e.g., the monograph by Mikhail Anisimov [18] and references there) have revealed a decisive role of the order-parameter fluctuations near the phase transition and critical (bifurcation) points.

As the theoretical background to study the properties of bounded systems, we use in our approach, the method of finding the pair correlation function (CF) $G_2(r)$ of the order parameter fluctuations, as a Green function of the Helmholtz operator, corresponding to

the differential Ornstein–Zernike (OZ) equation with appropriate boundary conditions at limiting surfaces [26,30].

The differential OZ equation is derived from the exact integral OZ equation for short-range direct correlation functions (DCF) $C(r)$ considering any number of its spatial moments C_i and short-range intermolecular potentials. Taking into account only the main contributions to CF $G_2(r)$ and the hydrophobic (zero) boundary conditions, the following formulae were obtained for $G_2(r)$ in the spatially confined systems with a geometry of slitlike and cylindrical pores [30]:

$$G_2(\rho^*, z) = (\pi H)^{-1} K_0[\rho^*(\kappa^2 + \pi^2/H^2)^{1/2}] \cos(\pi z/H), \quad (1)$$

$$G_2(\rho^*, z) = D_1 J_0(\mu_1 \rho^*/a_0) \exp[-(\kappa^2 + \mu_1^2/a_0^2)^{1/2}|z|]. \quad (2)$$

where $K_0(u)$, $J_0(u)$ are the McDonald and Besel functions, correspondingly;.$\rho^* = (x^2 + y^2)^{1/2}$; $\kappa = \xi^{-1}$ is the inverse value of the correlation length ξ;$\mu_1 \approx 2.405$ is the first zero of the Besel function; a_0 is the radius of the cylinder; H is the thickness of the slitlike pore.

Since CF $G_2(r)$ for the confined systems does not have an exponential form, it is natural to define the correlation length ξ of the order-parameter fluctuations, as a normalized second spatial moment M_2, in accordance to the following formula:

$$\xi = \sqrt{M_2} = \sqrt{\int G_2(r) r^2 dr / \int G_2(r) dr}. \quad (3)$$

Using this approach, the following formula was obtained for the deviation of the temperature variable from its critical value for the bounded systems:

$$\tau(S, \xi^*) = (G/S)^{1/\nu} + [1 + (G/S)^{1/\nu}](\xi^*)^{-1/\nu} sign\tau, \quad (4)$$

as well as similar formulas for the deviation of density $\Delta\rho(S, \xi^*)$ and pressure $\Delta P(S, \xi^*)$ from their critical values for spatially limited systems. In Formula (4) for the temperature deviation $\tau(S, \xi^*)$, the following designations were used: S is the number of monomolecular layers in the direction of the system confinement; $\xi^* = \xi/\xi_0$ is the dimensionless correlation radius with the amplitude ξ_0 of the correlation radius; G is a constant characterizing the geometric shape of a limited volume ($G = \pi$ for a slitlike pore, $G = \mu_1 = 2.405$ for a cylindrical sample, where μ_1 is the first zero of the Bessel function $J_0(z)$.

Direct consequences of Formula (4) are the following limiting cases: (a) for bulk systems when $S \gg \xi^*$, we obtain $\tau = (\xi^*)^{-1/\nu}$, which gives the well-known scaling formula for the temperature dependence of the correlation radius $\xi(\tau) = \xi_0 \tau^{-\nu}$; (b) for the bounded systems when $S \ll \xi^*$, instead of the dependence of the physical properties on the correlation radius, and through it on the thermodynamic variables, we obtain the dependence of these properties on S, which determines the size effects in accordance with the scaling formula $\tau(S) = (G/S)^{1/\nu}$.

As an example, consider the surface tension coefficient $\sigma(S, \xi^*)$, which, in accordance with (4) at $d = 3$, is described by the following formulas [43]:

$$(a)\ \sigma(\tau) = \sigma_0[\xi^*(\tau)]^{-2} \sim \tau^{2\nu}; \quad (5)$$

for bulk systems, when $S >> \xi^*$;

$$(b)\ \sigma(S) = \sigma_0[\tau(S)]^{2\nu} \sim 1/S^2 \quad (6)$$

for bounded systems, when $S \ll \xi^*$. The last result (6), which predicts the inversely proportional dependence of the surface tension coefficient $\sigma(S)$ on the square of the number S of the monomolecular layers in the direction of the spatial limitations of the volume under study, is new and needs to be experimentally verified.

3. Effective Critical Indices, Effective Fractal and Spatial Dimensions at the 2d-3d Crossover

Here, we consider the result, which is important from the point of view of the subject matter of this article, of the influence of the 2d-3d crossover on the effective fractal and spatial dimensions and on the effective values of the critical indices of bulk and bounded systems [41,42]. The problem discussed is as follows: how do the results of 3d systems transfer to the results of 2d systems, and vice versa? Obviously, such a 2d-3d C should be smooth and without discontinuities. The fact that the critical exponents in the 3d and 2d systems have different numerical values [18] and the results of computer experiments [28,29] should be taken into account while describing this DC.

The Kawasaki idea from the theory of mode coupling [23] was used to obtain the following formula for any effective critical exponents n giving a smooth transition from its 3d value n_3 to its 2d value n_2 [41,42]:

$$n = n_3 + \left\{ \frac{2}{\pi} \arctan(ax - b) - 1 \right\} \frac{n_3 - n_2}{2}. \quad (7)$$

Here $x = L/L_0$ is the dimensionless width of the slitlike pore or radius of the cylindrical pore; L_0 is the linear size of the system in the restricted geometry at which the crossover occurs; a and b are the dimensionless parameters characterizing slope and position of the 2d-3d crossover. In order to check Formula (7) by the results of the computer experiments [28,29], the size dependence of the critical exponent $\nu(H)$ for the correlation length was substituted into the following formula (see, for example, [28,41,42]):

$$\frac{T_c(H)}{T_c(\infty)} = 1 + kH^{-1/\nu} \quad (8)$$

where k is the coefficient of proportionality, H is the width of the slitlike pore. Formula (8) describes the shift of the critical temperature, $T_c^{pore} = T_c(H)$ compared with its bulk value $T^{3D} = T_c(\infty)$. The size dependence of the critical temperature $T_c(H)$ in the slitlike pores was theoretically obtained from (8) and shown in Figure 1, taking into account the dependence of the critical index ν on the number S of the monomolecular layers (see Table 1). As pointed out by the authors of [28] and that can be seen in Figure 1, the beginning of the 2d-3d C occurs approximately at a slitlike thickness $H_{cross} \approx 2.4$ nm. It corresponds to approximately eight monolayers of water molecules in the slitlike pore.

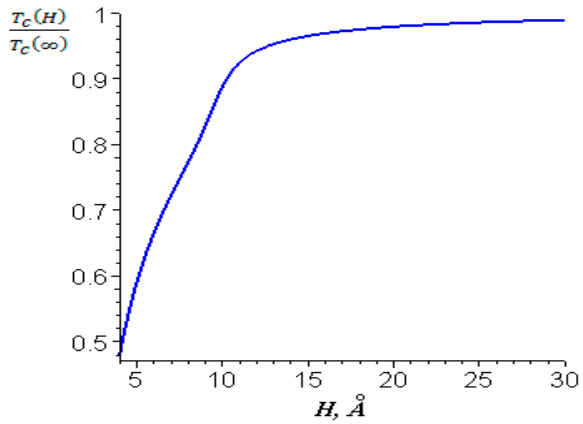

Figure 1. Size dependence of the critical temperature in the slitlike pore (finite-size scaling + Formula (7) for ν.

Table 1. Effective fractal, spatial dimensions and the critical exponents.

S	α_{eff}	β_{eff}	δ_{eff}	γ_{eff}	ν_{eff}	$d_{eff}=(2-\alpha_{eff})/\nu_{eff}$	$d_{fr}=d_{eff}-\beta_{eff}/\nu_{eff}$
1	0	0.125	15.000	1.750	1.000	2.000	1.875
2	0.025	0.171	10.544	1.632	0.915	2.158	1.971
3	0.026	0.173	10.416	1.629	0.912	2.163	1.973
4	0.027	0.174	10.339	1.625	0.910	2.168	1.977
5	0.028	0.176	10.199	1.619	0.906	2.177	1.982
6	0.030	0.179	10.000	1.611	0.900	2.189	1.990
7	0.032	0.184	9.696	1.600	0.892	2.206	2.000
8	0.036	0.191	9.277	1.581	0.878	2.237	2.019
9	0.044	0.205	8.546	1.547	0.853	2.293	2.053
10	0.059	0.233	7.335	1.476	0.802	2.420	2.129
11	0.084	0.278	5.892	1.360	0.719	2.665	2.278
12	0.102	0.313	5.064	1.272	0.655	2.898	2.742
13	0.110	0.3265	4.789	1.237	0.630	3.000	2.482

Table 1 contains the calculation results [42] of the effective critical exponents $\alpha_{eff}, \beta_{eff}, \delta_{eff}, \gamma_{eff}, \nu_{eff}$ for the confined fluids, taking into account a 2d↔3d crossover between the values of the critical exponents $\alpha = 0, \beta = 1/8, \delta = 15, \gamma = 7/4, \nu = 1$ for the 2d-systems and $\alpha = 0.110$, $\beta = 0.3265$, $\delta = 4.789$, $\gamma = 1.237$, $\nu = 0.630$ for the 3d-systems. These exact value data and theoretical calculations seem to be quite good. It should be stressed that the critical exponents for the 3d systems, obtained by M.A. Anisimov with his co-authors in [36–39], are now considered generally accepted for the systems belonging to the universality class of the 3d Ising model.

The last two columns in Table 1 contain the effective fractal d_{fr} and spatial d_{eff} dimensions calculated from a hyperscaling equality

$$d_{eff} = (2 - \alpha_{eff})/\nu_{eff} \tag{9}$$

and the B. Mandelbrot's formula

$$d_{fr} = d_{eff} - \beta_{eff}/\nu_{eff} \tag{10}$$

Figure 2 illustrates an interesting consequence of the DC in the process of layer ordering. Here, a smooth change can be seen in the effective spatial d_{eff} and fractal d_{fr} dimensions with a change in the number S of molecular layers.

As is known, the hyperscaling equalities between the critical indices are sometimes identified as such equalities that contain the spatial dimension d. In this sense, the well-known formula of Benoit Mandelbrot (10) for the fractal dimension d_{fr}, is also a hyperscaling equality, since it includes the spatial dimension d and the critical indices β and ν, which describe two main scaling laws, namely: the shape of the phase coexistence curve (binodal) and temperature dependence of the correlation radius, respectively.

It should also be noted that in layer-by-layer ordering, for each number of molecular layers, the scaling and hypercaling equalities, such as the Mandelbrot Formula (10), are valid. Therefore, one of the consequences of the results of calculating the fractal and spatial dimensions, as well as the critical exponents, is the following conclusion: *a 2d↔3d crossover leads to the gradually varying dependence of d_{eff} and d_{fr} on number S, which determines a fixed number of monolayers in the direction of the system's spatiality.*

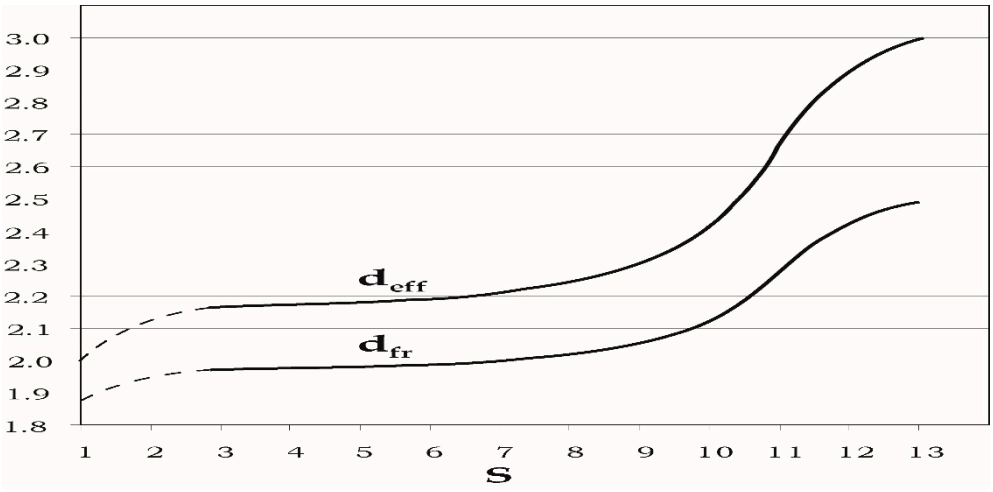

Figure 2. Effective fractal d_{fr} and spatial d_{eff} dimensions as a function of S..

Figure 2 illustrates the result of the size dependence of the fractal d_{fr} and the spatial d_{eff} dimensions on number S of the monomolecular layers obtained in Table 1.

It should be noted that a 2d-3d crossover occurs in confined liquids with a slitlike pore geometry filled with water, if the number of molecular layers $S \approx 8$. Actually, as was mentioned above, Brovchenko and Oleinikova [28,29], using results of their computer simulation, showed that the critical index ν of the temperature dependence of the correlation length changes from its bulk three-dimensional value ν = 0.63 to the two-dimensional value ν = 1.0 for water in the slitlike pores at a pore's thickness of about 2.4 nm.

4. Fractal Dimension, Its Connection with the Random Mean-Square Fluctuation of Order Parameters and a New Phase Formation

At the initial stages, new phase nuclei do not have a spherical shape but are fractal objects. An important characteristic of fractals is their fractal dimension d_{fr}, which does not coincide with the spatial dimension d (always $d_{fr} < d$). It turns out that the fluctuation of the number of particles in the nucleus of the new phase is directly determined by the fractal dimension of this cluster (see review article [48]).

Indeed, near the critical (bifurcation) points, the susceptibility χ of the system (e.g., the isothermal compressibility β_T of the liquid) becomes a strongly fluctuating quantity. As a result, the RMS fluctuation of the number of particles (in the general case—fluctuations of the order parameter)

$$< \Delta N^2 >^{1/2} \sim ((N^2/V)k_B T\chi)^{1/2} \sim L^{d/2}\rho(k_B T\chi)^{1/2} \quad (11)$$

reveals a rather complex dependence on the linear size L of the cluster. Here in (8), N is the number of particles in volume V, $\rho = N/V$ is the numerical density of the particles, k_B is the Boltzmann constant, T is the absolute temperature.

Far from the critical (bifurcation) points, the RMS fluctuation of the number of particles $< \Delta N^2 >^{1/2} \sim L^{d/2}$, for example, is determined only by the spatial dimension d of the system. It should be emphasized that this dependence changes radically when the fluctuation effects are taken into account. Due to the scaling hypothesis for the spatially limited systems [25] (the new phase nuclei are just such objects), the following formula for the susceptibility χ, as a fluctuating quantity, can be written in a bounded system:

$$\chi = L^{\gamma/\nu} f_\chi(A\tau L^{1/\nu}, BhL^{\beta/\nu}), \quad (12)$$

where $f_\chi(x, y)$ is the scaling function; $x = A\tau L^{1/\nu}$ is the first scaling argument connecting with the temperature variable $\tau = (T - T_c)/T_c$; $y = BhL^{\beta/\nu}$ is the second scaling argument connecting with the field variable h, which determines the influence of an external field on

the process of the new phase formation; A and B are the non-universal constants; β, γ, ν are the critical indices.

Based on Formulas (11) and (12), we have the following expression for the RMS fluctuation in the number of particles for a fractal cluster with an arbitrary spatial dimension d for a fluctuation region in a close neighborhood of critical points and points of the second-order phase transitions:

$$< \Delta N^2 >^{1/2} = L^{(\gamma/2\nu)+d/2} f_N(x, y). \tag{13}$$

Let us consider in more detail the value of the exponent $(\gamma/2\nu) + d/2$ in dependence (13) for the RMS fluctuations in the number of particles for a fractal cluster. For this purpose, we use the Mandelbrot formula $d_{fr} = d - \beta/\nu$, as well as the following scaling and hyperscaling (containing the spatial dimension d) equalities for the critical exponents: $\alpha + 2\beta + \gamma = 2$, $d\nu = 2 - \alpha$. As a result of the obvious transformations, we obtain the following result for the exponent $(\gamma/2\nu) + d/2$:

$$\frac{\gamma}{2\nu} + \frac{d}{2} = \frac{2 - \alpha - 2\beta}{2\nu} + \frac{d}{2} = \frac{d\nu - 2\beta}{2\nu} + \frac{d}{2} = \frac{d}{2} - \frac{\beta}{\nu} + \frac{d}{2} = d - \frac{\beta}{\nu} = d_{fr} \tag{14}$$

Thus, the scale transformation RMS fluctuation in the number of particles in the nucleus of a new phase is characterized by a critical index, which is exactly equal to the fractal dimension d_{fr} of this cluster, which made it possible to obtain, apparently for the first time in [48], the following formula, which is of fundamental importance for the theory of fractals and physics of phase transitions and critical phenomena:

$$< \Delta N^2 >^{1/2} = L^{d_{fr}} f_N(x, y). \tag{15}$$

The conclusions regarding the fractal nature of the new phase nuclei near the critical (bifurcation) points and the points of the structural phase transitions are confirmed, not only by the direct calculations, based on Formulas (11)–(14), but also by computer simulation results [46]. It should be emphasized again, the importance of using the Mandelbrot formula as a hyperscaling equality between the fractal and spatial dimensions and the critical exponents, to derive Formula (15).

As a result of studying the process of growth of the fractal clusters for the systems with a scalar order parameter and a centrally symmetric potential of intermolecular interaction, it was shown [46] that the fractal nuclei of the new phase quite quickly reach a spherical shape and increase or decrease in size depending on whether they have reached the critical radius R_{crit} or not. If the radius of such a nucleus of a new phase is $R < R_{crit}$, then the process of fluctuation resorption turns out to be energetically favorable for such a nucleus of a subcritical size. If the opposite inequality $R > R_{crit}$ holds, then the coalescence process is energetically favorable for such a supercritical nucleus of a new phase, in which the radius of the nucleus R begins to grow in time t, according to law $R \sim t^{1/3}$ obtained by I.M. Lifshits and V.V. Slezov [49].

5. Medical Consequences of Carcinogenesis and Nucleation Isomorphism

The physical mechanism of the formation (nucleation) of new phase nuclei considered above, is synergistically similar (isomorphic) to the process of carcinogenesis in oncology. In this section, we shall consider more rigorously the main medical consequences of the physical process of the formation of embryos of a new phase, obtained in theoretical studies and confirmed by the results of experiments and simulations.

The starting point of studying the process of nucleation of a new phase in a liquid-vapor metastable system, is the following formula for the minimum work (energy) W to create a spherical new-phase nucleus of radius R [24,43,46,47]:

$$W(R) = -(4/3)\pi R^3 \delta\mu\delta\rho_{coex} + 4\pi\sigma R^2, \tag{16}$$

Here in (16), the first and second terms correspond, respectively, to the volume and surface contributions to the energy of nucleation, $\delta\mu$=$\mu(\rho,T) - \mu(\rho_{coex}(T)$ is the degree of metastable vapor supersaturation, $\delta\rho_{coex} = [\rho_{coex}(T) - \rho_c]/\rho_c$ is the dimensionless deviation of the density along the binodal (coexistence curve) and in the critical point, σ is the surface tension coefficient.

Taking into account an obvious extremum condition $dW(R)/dR = 0$, one may calculate the critical size R_{cr} and the nucleation energy (nucleation barrier) $W(R_{cr})$, required to create a critical nucleus of a new phase:

$$R_{cr} = 2\sigma/\delta\mu\delta\rho_{coex} \tag{17}$$

$$W(R_{cr}) = 16\pi\sigma^3/3(\delta\mu)^2(\delta\rho_{coex})^2 \tag{18}$$

Allowing for (17) and (18), it is convenient to represent the nucleation energy $W(R)$ by the following expansion in the powers of the deviation of the nucleus size R from its critical value R_{cr}:

$$W(R) = W(R_{cr}) + 4\pi\sigma(R - R_{cr})^2 - (4\pi/3)\delta\mu\delta\rho_{coex}(R - R_{cr})^3 \tag{19}$$

This expansion is accurate, since all of the derivatives $W^{(n)}(R_{cr}) = 0$ for $n \geq 4$.

As it follows from (17)–(19), away from the points (lines) of phase transitions and the boundary of the stability, the growth of R_{cr} and $W(R_{cr})$ occurs (a) at the increasing surface tension coefficient σ, as is seen from numerator in Formulas (17) and (18), as well as (b), by decreasing both values of the degree of supersaturation (or superheating) $\delta\mu$ and the order parameter $\delta\rho_{\text{bin}}$ on the binodal (coexistence curve), as is seen from the denominator in Formulas (17) and (18).

With the approaching critical (bifurcation) points, the following situation should be realized:

(a) The critical size R_{cr} of the embryo of the new phase should increase as the correlation length ξ of the order parameter fluctuations, i.e., R_{cr}~ξ. In other words, the temperature dependence of the critical size of the embryo is described by the formula R_{cr}~$|\tau|^{-\nu}$, where $\tau = (T - T_c)/T_c$. This means that the critical nucleus size R_{cr} arises, according to the hyperbolic law $R_{cr} \sim 1/|T - T_c|^{\nu}$ at approaching the critical temperature T_{c} with the critical exponent $\nu = 0.63$ in a 3-dimensional system with linear dimensions L far exceeding the correlation length ξ, for example, for $L >> \xi$. This result, regarding the temperature dependence of $R_{cr}(\tau)$, directly follows from (17) taking into account the following formulas of the fluctuation theory of the phase transitions [13,17,18]: $\sigma \sim|\tau|^{(d-1)\nu}$, $\delta\mu \sim|\tau|^{\beta\delta}$, $\delta\rho_{coex} \sim|\tau|^{\beta}$, $\beta(\delta + 1) = \gamma + 2\beta = 2 - \alpha = d\nu$;

(b) The energetic nucleation barrier $W(R_{cr})$ appears to have no singularities at the critical (bifurcation) points or at the boundary of stability (spinodal) because Formula (18) can be rewritten in such an equivalent form:

$$W(R_{cr}) = 4\pi\sigma R_{cr}^2/3 \tag{20}$$

which gives the following result for the 3-dimensional case:

$$W(R_{cr}) \sim \sigma_{\text{cr}}2 \sim \sigma\xi^2 \sim \tau^{(d-1)\nu}\tau^{-2\nu} \sim \tau^0 \to const. \tag{21}$$

Now let us pay special attention to the role of the fluctuation effects near the critical (bifurcation) points. These effects can be estimated by the Ginsburg number Gi and by the Levanyuk–Ginsburg criterion [15,16] of the validity of the Landau thermodynamic theory of phase transitions [13]. The Ginsburg number Gi is determined as the ratio of the RMS fluctuation $<\Delta\phi^2>$ of the order parameter to the square of the equilibrium order parameter

ϕ_0^2. It is possible to show (see e.g., [17,18,24]) that the Ginsburg number is determined by the following formula:

$$Gi = < \Delta\phi^2 > / \phi_0^2 \sim \phi_0^{d-4} / \xi_0^d, \quad (22)$$

where ξ_0 is the amplitude of the correlation radius of the order parameter, which has the order of the radius of the intermolecular interaction (for water $\xi_0 \approx 0.3$ nm).

It follows that the Ginsburg number is rather small $Gi << 1$, near the critical (bifurcation) points and the boundaries of stability (spinodal), if the spatial dimensionality $d \geq 4$ and/or the intermolecular interaction is long-ranged, i.e., the value of the interaction radius becomes large enough ($\xi_0 \to \infty$). As a result, the fluctuation effects can be neglected for systems with a small Ginsburg number. In this case, the Landau theory of the phase transitions in the average-field approximation becomes valid.

Thus, the process of formation of a new phase is energetically disadvantageous along the coexistence curve (binodal) of the phases, and even for the deep quenching into the metastable region only for systems with a small Ginsburg number (for example, in a liquid-crystal state, which is typical for the medical and biological objects because of its membrane structures, in polymer mixtures with a long-ranged intermolecular interaction, etc.). The latter result is confirmed by the considerations related to the ratio (22) that gives the reverse-proportional dependence of the nucleation (carcinogenesis) barrier with the Ginsburg number *Gi* [24,27]:

$$W(R_{cr}) \sim Gi^{-1} \quad (23)$$

As was shown above, that the energy, which is necessary to create a new phase nucleus (tumor) or, similarly, height of the nucleation (carcinogenesis) barrier, becomes larger the smaller the Ginsburg number, therefore preventing the growth of the pathological neoplasm.

Here, an approach, based on using the hypothesis of a synergetic isomorphism (similarity) between the processes of carcinogenesis and nucleation allows for formulating the following conclusions:

1. The critical size of a tumor's fractal embryo is directly proportional to the coefficient of the surface tension.
2. The value of the energetic barrier of carcinogenesis is directly proportional to the cube of the surface tension coefficient and inverse proportional to the Ginsburg number *Gi*.
3. Our main fundamental medical finding is as follows: a necessary condition that should prevent the formation and further uncontrolled growth of pathological tumors, is the use the appropriate surface-inactive substances (surfinactants), which increase the surface tension coefficient.

As a result, this will lead to the following consequences:

(a) the increase in the value of the critical size of new phase fractal nuclei, which will reduce the number of subcritical embryos, thus decreasing their sizes,
(b) the creation of a higher nucleation barrier $W(R_{cr})$, which significantly reduces the probability of the emergence of capable fractal embryos and the further growth of such supercritical fractal nuclei.

The confirmation of the theoretical consequences, based on the hypothesis of the synergetic similarity between carcinogenesis and the nucleation processes requires, without any doubt, further experimental studies and careful selection of effective non-toxic surface-inactive substances. At the same time, we would like to believe that the attractive idea of a synergetic isomorphism of similar phenomena in open non-equilibrium systems of a different nature, which has already proved its effectiveness in a large number of studies, will be useful for a deeper understanding of the carcinogenesis processes and preventing the formation of cancer tumors.

The studies carried out allow us to hope that the theoretical results concerning the effects of 2d-3d C on the fractal dimension and other physical properties in various systems under confinement will stimulate new experimental studies and practical applications in this important direction.

6. Conclusions

This article is aimed at studying the effective fractal dimension of bulk and bounded condensed systems, taking into account the 2d-3d crossover effects. As a result of the study, the following conclusions are proposed:

1. For the numerical calculations of the effective fractal dimension d_{fr}, the Mandelbrot Formula (10) was used as a hyperscaling equation that includes, in addition to d_{fr}, the spatial dimension d and the critical exponents β and ν, which describe, respectively, the shape of the phase coexistence curve (binodal) and the temperature dependence of the correlation length of the order-parameter fluctuations. In the process of layer-by-layer ordering, it was assumed that all known scaling and hyperscaling equations, including the Mandelbrot formula, are satisfied for each fixed number S of the monomolecular layers. It has been established (see Table 1) that the 2d-3d crossover leads to a smooth change in the dependence of the effective fractal dimension d_{fr} on number S in the following interval: from $d_{fr} = 1.971$ at $d = 2$ (two-dimensional systems) to $d_{fr} = 2.482$ at $d = 3$ (three-dimensional systems),
2. It is proved that the size dependence of the RMS fluctuations of the particle (molecule) number for a fractal cluster is characterized by a critical index, which is exactly equal to the fractal dimension d_{fr} (see the derivation of Formula (15)),
3. Using the hypothesis of the synergetic similarity of the process of carcinogenesis in oncology and the process of nucleation in physics, the following results were obtained: (a) the critical size of a fractal tumor's embryo is directly proportional to the surface tension coefficient (see Formula (17)); (b) the energetic barrier of carcinogenesis is directly proportional to the cube of the surface tension coefficient (see Formula (17)) and the inverse proportional to the Ginzburg number (see Formula (23)). The immediate consequence of these results is the following fundamental conclusion: *the use of non-toxic and surface-inactive substances that increase the surface tension coefficient should prevent the emergence and uncontrolled growth of malignant neoplasms or, in other words, not promote the transition of the carcinogenesis process to the coalescence stage.* Thus, the apparent effectiveness of the science of fractals and other natural sciences (above all—physics) in the sciences of the animate world, will stimulate the development of modern medicine and medicine of the future.

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